Tables of Chebyshev Impedance-Transforming Networks of Low-Pass Filter Form

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Summary—**Tables of element values are presented for lumped-element Chebyshev impedance-transforming networks, as are tables giving the Chebyshev passband ripple of each design. These circuits consist of a ladder network formed using series inductances and shunt capacitances; they give a Chebyshev impedance match between resistor terminations of arbitrary ratio (designs with resistor termination ratios from 1.5 to 50 are tabulated). The responses of these networks have moderately high attenuation at dc (the amount of attenuation depends on the termination ratio); their attenuation then falls to a very low level in the Chebyshev operating band, and then rises steeply above the operating band in a manner typical of low-pass filters. Designs having operating-band fractional bandwidths ranging from 0.2 to 1.0 are given. These impedance-transforming networks can be realized in lumped-element form for low-frequency applications, and in semilumped-element form (such as corrugated waveguide) at microwave frequencies.**

General

NUMEROUS papers have appeared on the design of quarter-wave step-transformer structures. Among these papers is one by Young, which contains tables of designs for quarter-wave transformers having Chebyshev transmission characteristics.[1] Since the accurate calculation of such transformer designs can be very tedious, the existence of extensive tables of step-transformer designs has proved very valuable. Such transformers are extremely useful at microwave frequencies, but for applications involving frequencies much below 1000 Mc, the size of step transformers can become impractically large.

Herein tables of element values are presented for a form of lumped-element low-pass filter structure that has impedance-transforming properties similar to those of a step transformer but which can be constructed in very compact form, even at low frequencies. Though it is anticipated that this type of impedance-transforming structure will find greatest application at frequencies below the microwave range, it will be seen that the tabulated designs presented here will also be useful as prototypes for the design of semilumped-element impedance-transforming networks at microwave frequencies.

Fig. 1 shows the general form of the impedance-transforming structures under consideration. It should be noted that the structure is of the form of a conventional low-pass filter structure. The main difference between these structures and those of conventional low-pass filters is that conventional low-pass filters have terminating resistors of equal (or nearly equal) sizes at each end. In the case of the filters discussed here, the terminating resistors may be of radically different size, which means there will be a sizable reflection loss at zero frequency. As a result of this sizable attenuation $L_{A\mathrm{dc}}$ at zero frequency, the transmission characteristics of Chebyshev filters of this type have the form shown in Fig. 2. Note that there is a band of Chebyshev ripple extending from ω_a' to ω_b', and that above ω_b' the attenuation rises steeply in a manner typical of low-pass filter structures. It should be noted that the attenuation L_A indicated in Fig. 2 is transducer attenuation expressed in decibels; *i.e.*, it is the ratio of the available power of the generator to the power delivered to the load, expressed in db.

As mentioned above, the designs tabulated herein should prove useful for semilumped-element microwave structures, as well as for lumped-element low-frequency structures. An example of a semilumped-element waveguide structure is shown in Fig. 3. The structure shown is of the corrugated waveguide filter form[2,3] in which steps or corrugations in the guide height are used to simulate the shunt capacitors and series inductors of the structures in Fig. 1. In Fig. 3, where the guide top and bottom walls come close together, the effect is largely like that of a shunt capacitor (as is suggested by the dashed-lined capacitors in the figure). Where the guide top and bottom walls go far apart to form a groove, the effect is predominantly like that of a series inductance (as is suggested by the dashed-line inductance in the figure). In this manner, semilumped-element impedance-transforming structures can be designed from the prototype designs given herein. A structure such as that in Fig. 3 might prove desirable for waveguide applications, say at L-band or below, where a conventional step transformer might be larger than can be accommodated for some given application. Such structures can also find application at higher frequencies for use as structures for coupling to magnetically tunable yttrium-iron-garnet ferrimagnetic resonators. Structures of the form in Fig. 3 have advantages for such ap-

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[1] Leo Young, "Tables of cascaded homogeneous quarter-wave transformers," IRE Trans. on Microwave Theory and Technique, vol. MTT-7, pp. 233–237; April, 1959.

[2] S. B. Cohn, "Analysis of a wide-band waveguide filter," Proc. IRE, vol. 37, pp. 651–656; June, 1949.

[3] G. L. Matthaei, Leo Young, and E. M. T. Jones, "Design of Microwave Filters, Impedance-Matching Networks, and Coupling Structures," Stanford Research Inst., Menlo Park, Calif., prepared on SRI Project 3527, Contract DA 36-039 SC-87398, ch. 7; 1963.

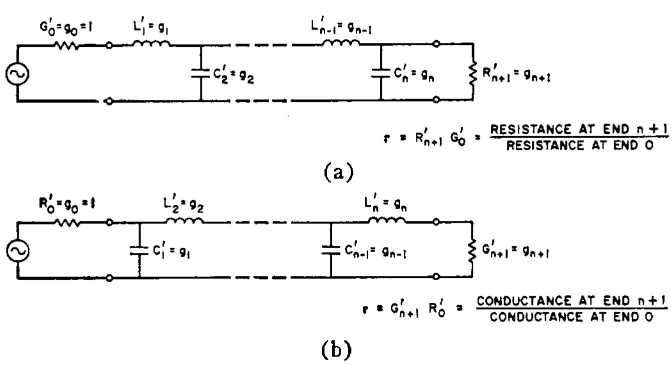

Fig. 1—Definition for normalized prototype element values for impedance-transforming networks of low-pass filter form. (The tabulated element values are normalized so that $g_0=1$ and $\omega_m'=1$. [See Fig. 2.])

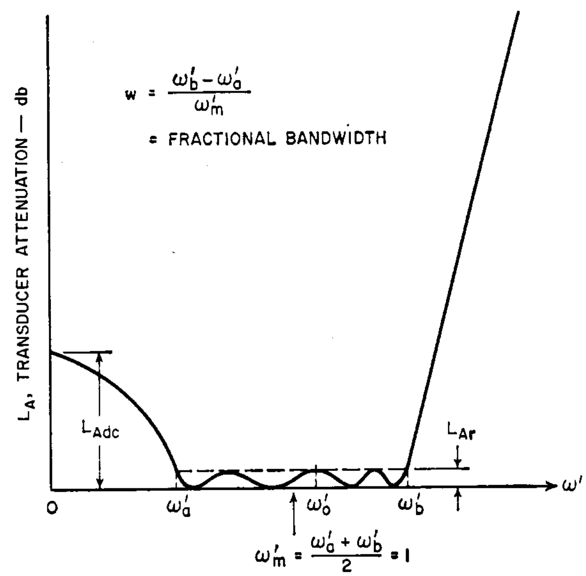

Fig. 2—Definition of response parameters for low-pass impedance-transforming filters. (The frequency scale for the tabulated prototype design is normalized so that $\omega_m'=1$, as indicated above.)

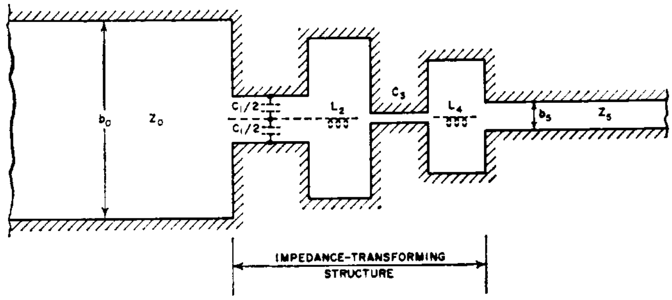

Fig. 3—A corrugated-waveguide structure for matching between guides of different heights.

plications because the inductive section at the low-impedance end of such a structure (forming L_4 at the right end in Fig. 3) provides needed room for locating a ferrimagnetic resonator while at the same time giving the impedance transformation required for obtaining tight coupling to the ferrimagnetic resonator.[4]

Parameters of the Attenuation Characteristics

The frequency scale of the networks tabulated here has been normalized as indicated in Fig. 2 so that the arithmetic mean-operating-band radian frequency,

$$\omega_m' = \frac{\omega_a' + \omega_b'}{2}, \tag{1}$$

is scaled so that

$$\omega_m' = 1. \tag{2}$$

Herein the frequency variables and element values of the normalized prototype circuits will be primed to indicate that they are normalized, and corresponding unprimed quantities will be reserved for the same parameters scaled to suit specific applications. With the normalization in (2),

$$w = \frac{\omega_b' - \omega_a'}{\omega_m'} = \omega_b' - \omega_a', \tag{3}$$

$$\omega_a' = 1 - \frac{w}{2}, \tag{4}$$

and

$$\omega_b' = 1 + \frac{w}{2} \cdot \tag{5}$$

In most impedance-transformer applications, the parameters of the normalized prototype circuit that will be of importance are the impedance or admittance transformation ratio r (see Fig. 1), the fractional bandwidth w, and the db passband attenuation ripple L_{Ar} (see Fig. 2). After a designer has determined the value of r, the minimum value of w, and the maximum allowable value of L_{Ar} for his application, the next step is to determine the number n of reactive elements required in the circuit in order to meet these specifications. The required value of n can easily be determined with the aid of Tables 1 to 5 (pp. 944–948). For example, suppose a designer desires an impedance transformer to give an $r=3$ impedance ratio over the band from 500 to 1000 Mc with 0.10-db or smaller attenuation ripple ratio in the operating band. The required fractional bandwidth is given by

$$w = \frac{f_b - f_a}{f_m} = \frac{2(f_b - f_a)}{f_b + f_a}, \tag{6}$$

[4] G. L. Matthaei, *et al.*, "Microwave Filters and Coupling Structures," Stanford Research Inst., Menlo Park, Calif., Final Rept., SRI Project 3527, Contract DA 36-039 SC-87398, Sec. III; February, 1963.

which for this example gives

$$= \frac{2(1000 - 500)}{1000 + 500} = 0.667.$$

This value of fractional bandwidth lies between the $w=0.6$ and $w=0.8$ values in Tables 1 to 5, so the $w=0.8$ value will be used. This will give an operating bandwidth somewhat larger than is actually required, which is often desirable. However, if this is objectionable, the desired bandwidth can be achieved by interpolating between the values in Tables 1 to 5 in order to determine the required value of n for $w=0.667$ and $L_{Ar} \leqq 0.10$ db, and then interpolating between numbers in the element-value tables (to be discussed later), to obtain a prototype with $w=0.667$, $r=3$, and the given value of n. Reducing the bandwidth from 0.8 to 0.667 would mean that a smaller value of L_{Ar} could be achieved for given values of r and n.

Assuming that use of $w=0.8$ is satisfactory, we determine the required value of n as follows. From Table 1 (which is for $n=2$ reactive elements), for $w=0.8$ and $r=3$, we obtain $L_{Ar}=0.639$ db, which is too large. From Table 2 (which is for $n=4$ reactive elements), for $w=0.8$ and $r=3$, we obtain $L_{Ar}=0.139$ db, which is still somewhat too large. Finally, from Table 3 we find that for $n=6$, $L_{Ar}=0.023$ db, which is less than the 0.10 db required. Actually, in this case it is clear that if the fractional bandwidth were reduced close to the $w=0.667$ minimum required value, $n=4$ reactive elements would be sufficient to give $L_{Ar}<0.10$ db.

Besides the fractional bandwidth w and the operating-band-ripple L_{Ar} in Fig. 2, other aspects of the response of the normalized prototype circuit will at times also be of interest. The attenuation $L_{A\text{dc}}$ at zero frequency is given by

$$L_{A\text{dc}} = 10 \log_{10} \frac{(r+1)^2}{4r} \text{ db} \tag{7}$$

where r is again the impedance or admittance transformation ratio.

In some cases it will be desired to determine the attenuation accurately over a range of frequencies, possibly for making use of the strong attenuation band of this type of structure above frequency ω_b'. The attenuation characteristic in Fig. 2 can be predicted by mapping the attenuation characteristic of a conventional Chebyshev low-pass filter (which has an equal-ripple pass band from $\omega''=0$ to $\omega''=\omega_1''$) by use of the mapping function

$$\frac{\omega''}{\omega_1''} = \frac{\omega'^2 - \omega_0'^2}{A} \tag{8}$$

where ω'' is the frequency variable for the conventional Chebyshev filter,

$$A = \frac{\omega_b'^2 - \omega_a'^2}{2}, \tag{9}$$

and

$$\omega_0' = \sqrt{\frac{\omega_a'^2 + \omega_b'^2}{2}} \tag{10}$$

is the frequency in the response in Fig. 2, which corresponds to $\omega''=0$ for the corresponding conventional Chebyshev low-pass filter characteristic. Making use of (8) and (9) and the equations for the attenuation of a conventional Chebyshev low-pass filter (see, for example, Fig. 2 of Cohn[5]), we obtain

$$L_A = 10 \log_{10} \left\{1 + \epsilon \cosh^2 \left[\frac{n}{2} \cosh^{-1}\left[\frac{2(\omega'^2 - \omega_0'^2)}{\omega_b'^2 - \omega_a'^2}\right]\right]\right\} \text{db}, \tag{11}$$

which applies in the "stop" bands $0 \leq \omega' \leq \omega_a'$ and $\omega_b' \leq \omega' \leq \infty$. In (11),

$$\epsilon = \left[\text{antilog}_{10} \frac{L_{Ar}}{10}\right] - 1, \tag{12}$$

and n is the number of reactive elements in the impedance-transforming filter network. In the operating band $\omega_a' \leq \omega' \leq \omega_b'$,

$$L_A = 10 \log_{10} \left\{1 + \epsilon \cos^2 \left[\frac{n}{2} \cos^{-1}\left[\frac{2(\omega'^2 - \omega_0'^2)}{\omega_b'^2 - \omega_a'^2}\right]\right]\right\} \text{db}. \tag{13}$$

A conventional low-pass Chebyshev response for a filter with n'' reactive elements maps into a response of the form in Fig. 2 for a filter having $n=2n''$ reactive elements. The response in Fig. 2 corresponds to an $n=8$ reactive-element design. However, this response could be obtained by mapping the response of a corresponding $n''=4$ reactive-element conventional Chebyshev filter design, using the mapping in (8). This is because the mapping function in (8) doubles the degree of the transfer function polynomial.

Tables of Prototype Element Values

Tables 6 to 10 (pp. 949–963) give element values for prototype impedance-transforming networks for $n=2$, 4, 6, 8, and 10 reactive elements. After the designer has arrived at values for r, w, and n, the normalized element values can be obtained from the tables. However, since the networks under discussion are antimetric[6] (*i.e.*, half of the network is the inverse of the other half), only half of the element values for each network are tabulated, and it is necessary to compute the element values of the second half of the network from those of the first half.

[5] S. B. Cohn, "Direct-coupled-resonator filters," Proc. IRE, vol. 45, pp. 187–196; February, 1957.

[6] E. A. Guillemin, "Synthesis of Passive Networks," John Wiley and Sons, Inc., New York, N. Y., ch. 11; 1957.

This procedure will now be summarized for each value of n. *For* $n=2$, $g_0=1$, and g_1 is obtained from Table 6. Then

$$g_2 = \frac{g_1}{r} \quad \text{and} \quad g_3 = r. \tag{14}$$

For $n=4$, $g_0=1$; and g_1 and g_2 are obtained from Tables 7(a), (b). Then

$$g_3 = g_2 r, \qquad g_4 = \frac{g_1}{r}, \quad \text{and} \quad g_5 = r. \tag{15}$$

For $n=6$, $g_0=1$; and g_1, g_2, and g_3 are obtained from Tables 8(a)–(c). Then

$$g_4 = \frac{g_3}{r}, \qquad g_5 = g_2 r,$$

$$g_6 = \frac{g_1}{r}, \quad \text{and} \quad g_7 = r. \tag{16}$$

For $n=8$, $g_0=1$; and g_1 to g_4 are obtained from Tables 9(a)–(d). Then

$$g_5 = g_4 r, \qquad g_6 = \frac{g_3}{r}, \qquad g_7 = g_2 r,$$

$$g_8 = \frac{g_1}{r}, \qquad g_9 = r. \tag{17}$$

For $n=10$, $g=1$; and g_1 to g_5 are obtained from Tables 10(a)–(e). Then

$$g_6 = \frac{g_5}{r}, \qquad g_7 = g_4 r, \qquad g_8 = \frac{g_3}{r},$$

$$g_9 = g_2 r, \qquad g_{10} = \frac{g_1}{r}, \qquad g_{11} = r. \tag{18}$$

Note that for a design with n reactive elements, there are $n+2$ element values, $g_0, g_1, \cdots, g_n, g_{n+1}$. Two possible interpretations of these element values are as indicated in Fig. 1. Observe that each of the terminating element values g_0 and g_{n+1} may be either a resistance or a conductance, depending on whether it is next to a shunt capacitance or a series inductance, respectively. This convention is used because it is convenient when applying the principle of duality to the element values. The two interpretations of the element values shown in Fig. 1 actually give the same final circuit for given terminations since the circuit shown at (b) could be obtained from that shown at (a) by simply turning the circuit at (a) around and scaling its impedance level.

In carrying out the calculations of the element values in Tables 6 to 10, eight significant figures were carried by the computer. However, the computations were made using a continued-fraction-expansion process,[6] which results in a gradual loss of accuracy as the expansion progresses. Thus, values for g_1 could be obtained with great accuracy, values for g_2 with somewhat less accuracy, etc. Tables 6 to 10 include all of the significant figures that were available in the computer printout; however, in the case of the values for, say, elements g_4 and g_5, some of the digits on the right end of each number are doubtless in error. In order to see how serious the errors might be, some responses were computed for several designs, including the rather extreme cases of $w=0.8$, and $w=1.0$, having $n=10$ and $r=50$. The bandwidths were as predicted, and after checking the details of the pass band, the pass band Chebyshev ripples were found to have very nearly the peak values predicted, with at most two or three units error in the third significant figure and better accuracy in most cases. Thus, though not all of the element values given in Tables 6 to 10 are as accurate as the number of significant figures shown indicates, the accuracy appears to be more than adequate for typical practical applications.

Scaling of the Normalized Design

After a designer has selected a normalized design, the element values required for his specific application are easily determined by scaling. Let R be the desired resistance level of one of the terminations, while R' is the corresponding resistance of the normalized design. Similarly, let $\omega_m=(\omega_a+\omega_b)/2$ be the *radian* frequency of the center of the desired operating band, while $\omega_m'=1$ is the corresponding frequency for the normalized design. Then the scaled element values are computed using

$$R_k = R_k'\left(\frac{R}{R'}\right) \tag{19}$$

$$C_k = C_k'\left(\frac{\omega_m'}{\omega_m}\right)\frac{R'}{R} \tag{20}$$

$$L_k = L_k'\frac{\omega_m'}{\omega_m}\ \frac{R}{R'} \tag{21}$$

where R_k', C_k', and L_k' are for the normalized design and R_k, C_k, and L_k are for the scaled design.

Derivation of the Tables

The circuit designs given in Tables 6 to 10 were synthesized by first starting with reflection coefficient functions for conventional Chebyshev low-pass filters that have pass bands from $\omega''=0$ to $\omega''=\omega_1''$, and have monotonically increasing attenuation above ω_1''. Reflection coefficient functions for such networks have the form[7]

[7] R. M. Fano, "A note on the solution of certain approximation problems in network synthesis," *J. Franklin Inst.*, vol. 249, pp. 189–205; March, 1950.

$$\Gamma(p'') = \frac{(p'' - p_{a1}'')(p'' - p_{a2}'')(p'' - p_{a3}'')(p'' - p_{a4}'') \cdots}{(p'' - p_{b1}'')(p'' - p_{b2}'')(p'' - p_{b3}'')(p'' - p_{b4}'') \cdots}, \tag{22}$$

where $p'' = \sigma'' + j\omega''$ is the complex frequency variable and the p_{ak}'' and p_{bk}'' give the locations of the zeros and poles of the function. Expressions giving the locations of these poles and zeros for a given circuit complexity and Chebyshev ripple amplitude can be found in the literature.[7] The poles and zeros of the reflection coefficient function were then mapped to give the reflection coefficient function[8]

$$\Gamma(p') = \frac{(p' - p_{a1}')(p' - \overline{p_{a1}'})(p' - p_{a2}')(p' - \overline{p_{a2}'}) \cdots}{(p' - p_{b1}')(p' - \overline{p_{b1}'})(p' - p_{b2}')(p' - \overline{p_{b2}'}) \cdots} \tag{23}$$

for the desired network. This was accomplished by use of (8) with ω'' replaced by p''/j, and ω' replaced by p'/j. The complex mapping function was then (with $\omega_1'' = 1$)

$$p'' = -j\left(\frac{p'^2 + \omega_0'^2}{A}\right). \tag{24}$$

By properly adapting (24) it can be seen that for a given pole location $p_{bk}'' = \sigma_{bk}'' + j\omega_{bk}''$ in the left half of the p'' plane, the corresponding pole locations p_{bk}' and $\overline{p_{bk}'}$ for the p' plane are given by

$$p_{bk}', \overline{p_{bk}'} = [\sqrt[4]{(\omega_0'^2 + A\omega_{bk}'')^2 + A(\sigma_{bk}'')^2}](\cos\theta \pm j\sin\theta) \tag{25}$$

where

$$\theta = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{A\sigma_{bk}''}{\omega_0'^2 + A\omega_{bk}''}\right). \tag{26}$$

Since σ_{bk}'' is negative, the argument of the arctangent function is usually negative so that the arctangent is evaluated as between 0 and $-\pi/2$ radians. However, since ω_{bk}'' is sometimes positive and sometimes negative, the argument of the arctangent can go positive in some cases. When the argument is positive, the arctangent is evaluated as having a value between $-\pi/2$ and $-\pi$ radians. In this manner, every pole in the $\Gamma(p'')$ reflection coefficient function yields two conjugate poles in the $\Gamma(p')$ function.

The mapping equations (25) and (26) also apply for mapping the zeros of the $\Gamma(p'')$ function to give those of the $\Gamma(p')$ function. The mapping of the zeros is somewhat simplified, however, since they lie on the imaginary axis in both planes, *i.e.*, $p_{ak}'' = j\omega_{ak}''$ and p_{ak}', $\overline{p_{ak}'} = \pm j\omega_{ak}'$.

[8] Here $\overline{p_{a1}'}$ is the complex conjugate of p_{a1}'.

We have not, as yet, discussed how the operating-band ripple enters into the synthesis problem. It can be shown that

$$L_{Ar} = 10\log_{10}(1 + \epsilon) \quad \text{db} \tag{27}$$

where

$$\epsilon = \frac{(r-1)^2}{4r\cosh^2\left[\frac{n}{2}\cosh^{-1}\left(\frac{\omega_0'^2}{A}\right)\right]} \tag{28}$$

is the same ϵ as appears in (11) to (13). Thus, L_{Ar} (and ϵ) are determined by $\omega_0'^2/A$ (which controls the fractional bandwidth w), by r, and by n. Next, we compute

$$a = \frac{\sinh^{-1}\sqrt{\frac{1}{\epsilon}}}{\frac{n}{2}} \tag{29}$$

and the parameter a is used in Fano's[7] equations for the pole and zero locations of (22). By specifying $n'' = n/2$ = [number of poles and number of zeros in (22)], ω_1'', and a, the function in (22) is entirely determined, with the aid of Fano's results.[7]

To summarize, the reflection coefficient function $\Gamma(p'')$, (22), for a conventional Chebyshev filter is obtained from Fano's paper, using (27) to (29) to fix the value of a to use in Fano's equations. Next, the poles and zeros of this function are mapped as discussed in connection with (25) and (26) in order to obtain the reflection coefficient function $\Gamma(p')$, (23), for the desired form of network. This gives $\Gamma(p')$ in factored form, and its factored polynomials must then be multiplied out so that the numerator and denominator are in unfactored form. Then, using standard methods of network synthesis,[6] the input impedance function $Z(p')$ is formed, and the element values g_k are obtained by expanding $Z(p')$ in a continued-fraction expansion.

Acknowledgment

The mapping in (8) to (10) for use in the synthesis problem described in this paper was suggested to this writer by Prof. H. J. Carlin of the Polytechnic Institute of Brooklyn, N. Y., M. H. Lawton of Stanford Research Institute, Menlo Park, Calif., prepared the IBM 7090 computer program used in making the calculations for the tables of impedance-matching network designs.

TABLE 1

L_{Ar} in db vs. w and r for n = 2

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	.001800	.007090	.015549	.026687	.054487	.085225	.114295
2.0	.005398	.021235	.046482	.079573	.161453	.250817	.334238
2.5	.009712	.038148	.083313	.142197	.286429	.441537	.584260
3.0	.014381	.056397	.122863	.209037	.417880	.639116	.839801
4.0	.024240	.094750	.205359	.347127	.683492	1.02961	1.33539
5.0	.034434	.134143	.289233	.485751	.942628	1.40036	1.79552
6.0	.044782	.173863	.372948	.622407	1.19134	1.74770	2.21849
8.0	.065671	.253235	.537751	.836733	1.65534	2.37619	2.96665
10.0	.086637	.331840	.697855	1.13795	2.07792	2.92951	3.60972
15.0	.138917	.523445	1.07618	1.71210	2.98813	4.07171	4.90052
20.0	.190748	.707556	1.42530	2.22074	3.74255	4.97794	5.89726
25.0	.242036	.884410	1.74884	2.67658	4.38591	5.72825	6.70839
30.0	.292759	1.05445	2.05014	3.08935	4.94650	6.36822	7.39203
40.0	.392511	1.37594	2.59671	3.81341	5.88881	7.42049	8.50279
50.0	.490055	1.67535	3.08228	4.43402	6.66291	8.26714	9.38680

TABLE 2

L_{A_r} in db vs. w and r for n = 4

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	.000005	.000072	.000366	.001154	.005765	.017581	.039890
2.0	.000014	.000217	.001098	.003462	.017273	.052530	.118586
2.5	.000024	.000391	.001976	.006229	.031042	.094102	.211177
3.0	.000036	.000579	.002928	.009226	.045910	.138693	.309306
4.0	.000061	.000977	.004939	.015557	.077193	.231536	.509855
5.0	.000087	.001389	.007023	.022109	.109378	.325710	.708365
6.0	.000113	.001809	.009142	.028765	.141885	.419483	.901453
8.0	.000166	.002659	.013432	.042219	.207003	.603465	1.26813
10.0	.000220	.003516	.017754	.055746	.271701	.781440	1.60901
15.0	.000355	.005670	.028605	.089577	.430265	1.19921	2.36390
20.0	.000490	.007830	.039465	.123258	.583748	1.58151	3.00880
25.0	.000625	.009993	.050312	.156725	.732186	1.93332	3.57090
30.0	.000761	.012155	.061140	.189958	.875812	2.25897	4.06884
40.0	.001032	.016479	.082725	.255704	1.14959	2.84532	4.92088
50.0	.001303	.020801	.104210	.320490	1.40739	3.36201	5.63309

TABLE 3

L_{A_r} in db vs. w and r for n = 6

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	.000000	.000001	.000008	.000046	.000527	.002940	.010951
2.0	.000000	.000002	.000025	.000139	.001580	.008813	.032769
2.5	.000000	.000004	.000044	.000250	.002844	.015850	.058808
3.0	.000000	.000006	.000066	.000371	.004213	.023462	.086841
4.0	.000000	.000010	.000111	.000625	.007107	.039518	.145553
5.0	.000000	.000014	.000158	.000889	.010105	.056097	.205574
6.0	.000000	.000018	.000206	.001158	.013153	.072901	.265808
8.0	.000000	.000027	.000303	.001702	.019321	.106746	.385332
10.0	.000001	.000035	.000401	.002250	.025532	.140614	.502630
15.0	.000001	.000057	.000646	.003630	.041114	.224636	.784325
20.0	.000001	.000078	.000893	.005013	.056692	.307336	1.04972
25.0	.000002	.000100	.001140	.006398	.072235	.388599	1.30015
30.0	.000002	.000122	.001387	.007784	.087732	.468422	1.53707
40.0	.000003	.000165	.001881	.010555	.118578	.623873	1.97556
50.0	.000003	.000209	.002375	.013325	.149217	.774001	2.37394

TABLE 4

L_{A_r} in db vs. w and r for n = 8

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	.000000	.000000	.000000	.000002	.000047	.000474	.002805
2.0	.000000	.000000	.000001	.000006	.000142	.001421	.008408
2.5	.000000	.000000	.000001	.000010	.000256	.002557	.015123
3.0	.000000	.000000	.000001	.000015	.000380	.003788	.022386
4.0	.000000	.000000	.000003	.000025	.000641	.006391	.037710
5.0	.000000	.000000	.000004	.000036	.000912	.009086	.053534
6.0	.000000	.000000	.000005	.000046	.001187	.011827	.069577
8.0	.000000	.000000	.000007	.000068	.001745	.017375	.101897
10.0	.000000	.000000	.000009	.000090	.002307	.022963	.134250
15.0	.000000	.000001	.000015	.000145	.003721	.036984	.214561
20.0	.000000	.000001	.000020	.000201	.005139	.051006	.293676
25.0	.000000	.000001	.000026	.000256	.006559	.065002	.371479
30.0	.000000	.000001	.000031	.000312	.007979	.078962	.447963
40.0	.000000	.000002	.000042	.000423	.010820	.106762	.597078
50.0	.000000	.000002	.000053	.000534	.013660	.134394	.741291

TABLE 5

L_{A_r} in db vs. w and r for n = 10

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	.000000	.000000	.000000	.000000	.000004	.000076	.000705
2.0	.000000	.000000	.000000	.000000	.000013	.000228	.002116
2.5	.000000	.000000	.000000	.000000	.000023	.000410	.003808
3.0	.000000	.000000	.000000	.000001	.000034	.000607	.005640
4.0	.000000	.000000	.000000	.000001	.000058	.001024	.009514
5.0	.000000	.000000	.000000	.000001	.000082	.001457	.013524
6.0	.000000	.000000	.000000	.000002	.000107	.001897	.017601
8.0	.000000	.000000	.000000	.000003	.000157	.002788	.025849
10.0	.000000	.000000	.000000	.000004	.000208	.003686	.034152
15.0	.000000	.000000	.000000	.000006	.000335	.005945	.054960
20.0	.000000	.000000	.000000	.000008	.000453	.008210	.075739
25.0	.000000	.000000	.000001	.000010	.000591	.010477	.096447
30.0	.000000	.000000	.000001	.000012	.000719	.012745	.117070
40.0	.000000	.000000	.000001	.000017	.000975	.017278	.158045
50.0	.000000	.000000	.000001	.000021	.001231	.021808	.198650

TABLE 6

ELEMENT VALUE g_1 vs. w and r for n = 2

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	.706219	.703597	.699283	.693375	.677285	.656532	.632455
2.0	.998752	.995037	.988936	.980581	.957826	.928477	.894427
2.5	1.22321	1.21867	1.21119	1.20096	1.17309	1.13715	1.09545
3.0	1.41245	1.40720	1.39857	1.38675	1.35457	1.31306	1.26491
4.0	1.72989	1.72345	1.71289	1.69842	1.65900	1.60817	1.54919
5.0	1.99750	1.99007	1.97787	1.96116	1.91565	1.85695	1.78885
6.0	2.23328	2.22497	2.21133	2.19265	2.14176	2.07614	2.00000
8.0	2.64245	2.63262	2.61648	2.59437	2.53417	2.45652	2.36643
10.0	2.99626	2.98511	2.96681	2.94174	2.87348	2.78543	2.68328
15.0	3.73699	3.72309	3.70026	3.66900	3.58386	3.47404	3.34664
20.0	4.35346	4.33727	4.31067	4.27425	4.17507	4.04714	3.89872
25.0	4.89287	4.87467	4.84478	4.80384	4.69237	4.54859	4.38178
30.0	5.37845	5.35844	5.32559	5.28059	5.15805	5.00000	4.81664
40.0	6.23721	6.21400	6.17590	6.12372	5.98162	5.79833	5.58570
50.0	6.99127	6.96526	6.92255	6.86406	6.70478	6.49934	6.26099

TABLE 7(a)

ELEMENT VALUE g_1 vs. w and r for n = 4

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	.654788	.657906	.660411	.663789	.672627	.682851	.691997
2.0	.817774	.821133	.825784	.832114	.849169	.870103	.891140
2.5	.930430	.934675	.941288	.950374	.975182	1.00639	1.03907
3.0	1.01869	1.02390	1.03237	1.04407	1.07628	1.11740	1.16137
4.0	1.15504	1.16246	1.17444	1.19107	1.23732	1.29733	1.36291
5.0	1.26113	1.27034	1.28561	1.30687	1.35637	1.44432	1.53036
6.0	1.34862	1.35977	1.37816	1.40383	1.47602	1.57115	1.67668
8.0	1.49002	1.50469	1.52897	1.56298	1.65926	1.78699	1.92906
10.0	1.60350	1.62135	1.65115	1.69304	1.81212	1.97059	2.14655
15.0	1.82011	1.84551	1.88814	1.94834	2.12059	2.34997	2.60193
20.0	1.98377	2.01613	2.07060	2.14780	2.36941	2.66341	2.98220
25.0	2.11734	2.15623	2.22189	2.31517	2.58343	2.93733	3.31624
30.0	2.23119	2.27632	2.35267	2.46138	2.77415	3.18416	3.61804
40.0	2.42036	2.47726	2.57388	2.71189	3.10874	3.62196	4.15411
50.0	2.57580	2.64380	2.75961	2.92539	3.40107	4.00809	4.62699

TABLE 7(b)

ELEMENT VALUE g_2 vs. w and r for n = 4

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	.879438	.877473	.873990	.869079	.854894	.834880	.809414
2.0	.864180	.861059	.856062	.849091	.829351	.802463	.769794
2.5	.832360	.828680	.822701	.814392	.791112	.760009	.723214
3.0	.801475	.797396	.790697	.781408	.755573	.721536	.682056
4.0	.749720	.744953	.737205	.726515	.697104	.659162	.616451
5.0	.709217	.704050	.695548	.683859	.651966	.611495	.567004
6.0	.676872	.671307	.662216	.649751	.615978	.573722	.528195
8.0	.627771	.621635	.611640	.598002	.561485	.516884	.470430
10.0	.591627	.585091	.574412	.559894	.521413	.475373	.428762
15.0	.530784	.523455	.511533	.495457	.453742	.405944	.360212
20.0	.491325	.483417	.470604	.453442	.409704	.361356	.317074
25.0	.462747	.454380	.440863	.422868	.377735	.329378	.286632
30.0	.440656	.431899	.417799	.399129	.352985	.304898	.263634
40.0	.407962	.398574	.383534	.363817	.316349	.269196	.230600
50.0	.384325	.374422	.358638	.338129	.289898	.243886	.207563

TABLE 8(a)

ELEMENT VALUE g_1 vs. w and r for n = 6

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	—	.552987	.557215	.561832	.574866	.593425	.617252
2.0	—	.653863	.660472	.657505	.688083	.718119	.757844
2.5	—	.721280	.728264	.737457	.764324	.804014	.857477
3.0	—	.771565	.779848	.790692	.823020	.871286	.937176
4.0	—	.846080	.856701	.870676	.912464	.975829	1.06404
5.0	—	.902357	.914421	.930946	.980935	1.05763	1.16593
6.0	—	.947434	.960850	.979755	1.03712	1.12596	1.25282
8.0	—	1.01812	1.03396	1.05694	1.12737	1.23812	1.39896
10.0	1.05045	1.07258	1.09104	1.11757	1.19957	1.33006	1.52196
15.0	1.14880	1.17336	1.19692	1.23102	1.33796	1.51205	1.77348
20.0	1.21970	1.24698	1.27462	1.31514	1.44358	1.65616	1.97959
25.0	1.28578	1.30547	1.33683	1.38303	1.53081	1.77864	2.15901
30.0	1.33103	1.35436	1.38912	1.44049	1.60610	1.88685	2.32038
40.0	1.40479	1.43383	1.47467	1.53532	1.73341	2.07503	2.60632
50.0	1.46949	1.49776	1.54390	1.61286	1.84043	2.23812	2.85856

TABLE 8(b)

ELEMENT VALUE g_2 vs. w and r for n = 6

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	——	.914105	.912243	.909237	.899970	.885529	.864663
2.0	——	.919270	.915637	.910661	.895647	.872746	.840686
2.5	——	.907151	.902547	.896131	.876950	.848018	.808201
3.0	——	.892603	.887151	.879664	.857242	.823664	.778019
4.0	——	.865323	.858546	.849355	.822056	.781591	.727560
5.0	——	.841914	.834376	.823963	.793043	.747552	.687633
6.0	——	.822226	.814080	.802648	.768852	.719431	.655072
8.0	——	.790690	.781607	.768642	.730435	.675097	.604384
10.0	.777424	.766442	.756431	.742301	.700754	.641043	.565949
15.0	.734308	.722852	.711334	.695104	.647619	.580413	.498635
20.0	.704615	.692621	.680110	.662389	.610760	.538591	.453152
25.0	.678244	.669738	.656413	.637532	.582714	.506915	.419321
30.0	.660959	.651419	.637408	.617576	.560166	.481562	.392684
40.0	.634147	.623217	.608099	.586752	.525275	.442579	.352607
50.0	.612003	.601909	.585945	.563407	.498795	.413256	.323259

TABLE 8(c)

ELEMENT VALUE g_3 vs. w and r for n = 6

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	———	1.39189	1.38976	1.38374	1.35621	1.34284	1.31464
2.0	———	1.65447	1.65327	1.64498	1.62203	1.59230	1.55747
2.5	———	1.86740	1.86238	1.85303	1.82656	1.79262	1.75362
3.0	———	2.04932	2.04397	2.03306	2.00373	1.96660	1.92458
4.0	———	2.35952	2.35437	2.34203	2.30826	2.26631	2.22007
5.0	———	2.62876	2.62128	2.60724	2.57000	2.52445	2.47534
6.0	———	2.86768	2.85826	2.84331	2.80320	2.75476	2.70358
8.0	———	3.28642	3.27353	3.25656	3.21176	3.15887	3.10500
10.0	3.57519	3.64755	3.63487	3.61616	3.56757	3.51135	3.45600
15.0	4.32771	4.40936	4.39459	4.37252	4.31659	4.25472	4.19843
20.0	4.95220	5.04625	5.02813	5.00353	4.94205	4.87671	4.82157
25.0	5.59978	5.60283	5.58267	5.55594	5.48997	5.42243	5.36952
30.0	6.08124	6.10339	6.08183	6.05317	5.98343	5.91452	5.86447
40.0	6.93498	6.98759	6.96401	6.93206	6.85623	6.78622	6.74289
50.0	7.75679	7.76402	7.73777	7.70307	7.62248	7.55279	7.51680

TABLE 9(a)

ELEMENT VALUE g_1 vs. w and r for n = 8

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	———	———	.455719	.478895	.493830	.514197	.542203
2.0	———	———	.540590	.551575	.572148	.602034	.644270
2.5	———	———	.587676	.598015	.622786	.660085	.713692
3.0	———	———	.619468	.632424	.660711	.704259	.767640
4.0	———	———	.669224	.682219	.716904	.770829	.850888
5.0	———	———	.705926	.718858	.758687	.821292	.915641
6.0	———	———	.732731	.747994	.792241	.862435	.969540
8.0	———	———	.776124	.793190	.844841	.928134	1.05775
10.0	———	———	.808962	.827842	.885852	.980421	1.12990
15.0	———	.841923	.868151	.891087	.962018	1.08018	1.27253
20.0	———	.892332	.910889	.936721	1.01614	1.15603	1.38542
25.0	———	.921226	.944430	.972853	1.06328	1.21858	1.48139
30.0	———	.945450	.972209	1.00293	1.10141	1.27247	1.56623
40.0	———	.990528	1.01691	1.05161	1.16422	1.36356	1.71387
50.0	———	1.02565	1.05226	1.09060	1.21549	1.44007	1.84195

TABLE 9(b)

ELEMENT VALUE g_2 vs. w and r for n = 8

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	———	———	.882869	.891826	.889498	.883542	.872284
2.0	———	———	.912871	.911377	.904206	.891146	.869198
2.5	———	———	.915840	.912537	.901833	.883320	.853352
3.0	———	———	.913547	.908941	.895461	.872552	.836225
4.0	———	———	.904478	.898503	.880613	.850889	.804866
5.0	———	———	.894460	.887791	.866523	.831597	.778309
6.0	———	———	.885792	.877901	.853892	.814763	.755688
8.0	———	———	.869951	.860801	.832512	.786810	.718858
10.0	———	———	.856885	.846673	.815024	.764262	.689624
15.0	———	.844824	.831899	.819587	.781817	.721899	.635568
20.0	———	.823043	.813252	.799578	.757420	.691015	.596777
25.0	———	.810364	.798502	.783709	.738122	.666687	.566579
30.0	———	.799705	.786299	.770576	.722151	.646603	.541899
40.0	———	.779936	.766816	.749599	.696624	.614591	.503073
50.0	———	.764694	.751629	.733139	.676585	.589533	.473173

TABLE 9(c)

ELEMENT VALUE g_3 vs. w and r for n = 8

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	——	——	1.25934	1.29612	1.29034	1.27892	1.26486
2.0	——	——	1.45645	1.46526	1.46007	1.45062	1.44004
2.5	——	——	1.58652	1.59125	1.58727	1.58113	1.57554
3.0	——	——	1.68450	1.69517	1.69278	1.69040	1.69026
4.0	——	——	1.85605	1.86295	1.86630	1.87163	1.88272
5.0	——	——	1.99783	2.00032	2.00900	2.02207	2.04432
6.0	——	——	2.10995	2.11831	2.13213	2.15274	2.18591
8.0	——	——	2.30736	2.31704	2.34035	2.37546	2.42966
10.0	——		2.47012	2.48253	2.51533	2.56420	2.63844
15.0	——	2.71634	2.79330	2.81425	2.86867	2.94937	3.07037
20.0	——	3.02604	3.05093	3.07750	3.15154	3.26139	3.42563
25.0	——	3.21799	3.26761	3.30014	3.39222	3.52927	3.73421
30.0	——	3.38721	3.45682	3.49501	3.60421	3.76700	4.01064
40.0	——	3.72255	3.77993	3.82877	3.96999	4.18089	4.49736
50.0	——	4.00261	4.05198	4.11234	4.28306	4.53870	4.92336

TABLE 9(d)

ELEMENT VALUE g_4 vs. w and r for n = 8

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	——	——	1.05874	1.03830	1.01836	.993310	.963217
2.0	——	——	.971922	.958291	.936070	.908264	.875128
2.5	——	——	.903029	.891491	.868833	.839925	.805666
3.0	——	——	.852023	.837866	.815200	.785765	.751047
4.0	——	——	.769428	.758262	.734992	.705234	.670348
5.0	——	——	.708982	.700545	.677174	.647431	.612734
6.0	——	——	.666138	.656196	.632871	.603277	.568883
8.0	——	——	.600315	.591261	.568250	.539051	.505343
10.0	——	——	.553993	.545193	.522369	.493595	.460546
15.0	——	.498537	.478837	.470035	.447747	.419926	.388285
20.0	——	.438467	.431239	.422849	.401032	.373988	.343462
25.0	——	.407607	.397601	.389399	.368007	.341613	.312003
30.0	——	.383624	.372057	.363998	.342952	.317110	.288273
40.0	——	.343258	.334996	.327154	.306662	.281728	.254148
50.0	——	.315316	.308895	.301070	.281038	.256829	.230252

TABLE 10(a)

ELEMENT VALUE g_1 vs. w and r for n = 10

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	—	—	—	.411731	.431053	.451413	.480211
2.0	—	—	—	.462862	.488315	.516495	.557240
2.5	—	—	—	.500630	.524248	.558216	.607974
3.0	—	—	—	.524950	.550795	.589321	.646531
4.0	—	—	—	.558062	.589262	.635197	.704635
5.0	—	—	—	.581406	.617386	.669225	.748736
6.0	—	—	—	.602333	.639613	.696516	.784761
8.0	—	—	—	.633330	.673939	.739295	.842469
10.0	—	—	—	.654910	.700257	.772676	.888593
15.0	—	—	—	.695335	.748246	.834819	.977172
20.0	—	—	—	.724629	.782878	.880788	1.04508
25.0	—	—	.717659	.746793	.810293	.917900	1.10145
30.0	—	—	.736946	.765100	.833166	.949350	1.15033
40.0	—	—	.766902	.794839	.870277	1.00139	1.23350
50.0	—	—	.789986	.818025	.900056	1.04409	1.30394

TABLE 10(b)

ELEMENT VALUE g_2 vs. w and r for n = 10

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	———	———	———	.851504	.858625	.860767	.859701
2.0	———	———	———	.881630	.884712	.881901	.872399
2.5	———	———	———	.894402	.892594	.885695	.869471
3.0	———	———	———	.898963	.894896	.884561	.862817
4.0	———	———	———	.901233	.893787	.877959	.847490
5.0	———	———	———	.900489	.890215	.870070	.832824
6.0	———	———	———	.898423	.886036	.862320	.819526
8.0	———	———	———	.893312	.877654	.848232	.796645
10.0	———	———	———	.888545	.869969	.836018	.777567
15.0	———	———	———	.877509	.853825	.811467	.740419
20.0	———	———	———	.868173	.840950	.792449	.712355
25.0	———	———	.872608	.860536	.830258	.776873	.689706
30.0	———	———	.866148	.853926	.821092	.763644	.670666
40.0	———	———	.855489	.842725	.805899	.741889	.639712
50.0	———	———	.846863	.833681	.793533	.724303	.614990

TABLE 10(c)

ELEMENT VALUE g_3 vs. w and r for n = 10

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	——	——	——	1.19672	1.20715	1.20487	1.20163
2.0	——	——	——	1.30620	1.32431	1.32685	1.33043
2.5	——	——	——	1.39716	1.40681	1.41462	1.42546
3.0	——	——	——	1.46132	1.47303	1.48563	1.50357
4.0	——	——	——	1.55648	1.57754	1.59944	1.63084
5.0	——	——	——	1.62930	1.66064	1.69095	1.73480
6.0	——	——	——	1.69879	1.73040	1.76863	1.82409
8.0	——	——	——	1.80918	1.84520	1.89779	1.97463
10.0	——	——	——	1.89137	1.93900	2.00463	2.10094
15.0	——	——	——	2.05722	2.12274	2.21670	2.35605
20.0	——	——	——	2.18713	2.26528	2.38365	2.56095
25.0	——	——	2.21549	2.29086	2.38381	2.52418	2.73599
30.0	——	——	2.31023	2.38007	2.48646	2.64705	2.89085
40.0	——	——	2.46527	2.53177	2.66020	2.85726	3.15972
50.0	——	——	2.59137	2.65589	2.80583	3.03575	3.39173

TABLE 10(d)

ELEMENT VALUE g_4 vs. w and r for n = 10

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	———	———	———	1.09058	1.06762	1.04110	1.00815
2.0	———	———	———	1.04576	1.01344	.981121	.941372
2.5	———	———	———	.994964	.965926	.930146	.886468
3.0	———	———	———	.956631	.926490	.888424	.842083
4.0	———	———	———	.899757	.865317	.824186	.774461
5.0	———	———	———	.857803	.819348	.776307	.724499
6.0	———	———	———	.819720	.783020	.738609	.685385
8.0	———	———	———	.763647	.728082	.681890	.626982
10.0	———	———	———	.725540	.687563	.640251	.584402
15.0	———	———	———	.657517	.618468	.569723	.512800
20.0	———	———	———	.611571	.572919	.523495	.466326
25.0	———	———	.607518	.578870	.539483	.489711	.432565
30.0	———	———	.578024	.553219	.513311	.463401	.406408
40.0	———	———	.535061	.514119	.474094	.424140	.367651
50.0	———	———	.504294	.485800	.445368	.395513	.339610

TABLE 10(e)

ELEMENT VALUE g_5 vs. w and r for n = 10

r \ w	0.1	0.2	0.3	0.4	0.6	0.8	1.0
1.5	———	———	———	1.48519	1.48356	1.45206	1.41460
2.0	———	———	———	1.71767	1.72645	1.68796	1.64303
2.5	———	———	———	1.94819	1.92620	1.88395	1.83404
3.0	———	———	———	2.13122	2.10366	2.05705	2.00302
4.0	———	———	———	2.43218	2.41118	2.35864	2.29794
5.0	———	———	———	2.68509	2.67837	2.62004	2.55395
6.0	———	———	———	2.94350	2.91608	2.85342	2.78261
8.0	———	———	———	3.38623	3.33351	3.26359	3.18567
10.0	———	———	———	3.74099	3.69725	3.62163	3.53800
15.0	———	———	———	4.51953	4.46468	4.37775	4.28198
20.0	———	———	———	5.18505	5.10747	5.00980	4.90576
25.0	———	———	5.52426	5.75020	5.67014	5.56385	5.45242
30.0	———	———	6.08252	6.25905	6.17621	6.06327	5.94495
40.0	———	———	7.05445	7.17062	7.07131	6.94604	6.81695
50.0	———	———	7.89711	7.95889	7.85669	7.72001	7.58314